IF-THEN RULES AND FUZZY INFERENCE

IF-THEN RULES
AND FUZZY INFERENCE

Representation of knowledge

- To perform inference, knowledge should be represented in some form

Representation of knowledge as rules is the most popular form.

if \( x \) is \( A \) then \( y \) is \( B \)
(where \( A \) and \( B \) are linguistic values defined by fuzzy sets on universes of discourse \( X \) and \( Y \)).

- \( A \) rule is also called a fuzzy implication
- \( x \) is \( A \)" is called the antecedent or premise
- \( y \) is \( B \)" is called the consequence or conclusion

Inference

\textbf{inference} \n. n. [From \textit{Infer.}]

1. The act or process of inferring by deduction or induction.
2. That which inferred; a truth or proposition drawn from another which is admitted or supposed to be true; a conclusion; a deduction. --Milton.

Inference is a process of obtaining new knowledge through existing knowledge.

Knowledge as Rules

- How do you reason?
  - You want to play golf on Saturday or Sunday and you don’t want to get wet when you play.
- Use rules!
  - If it rains, you get wet!
  - If you get wet, you can’t play golf
- If it rains on Saturday and won’t rain on Sunday
  - You play golf on Sunday!

- Knowledge is rules
- Rules are in black-and-white language
  - Bivalent rules
- AI has so far, after over 30 years of research, not produced smart machines!
  - Because they can’t yet put enough rules in the computer (use 100-1000 rules, need >100k)
  - Throwing more rules at the problem

*Fuzzy Thinking: The new Science of Fuzzy Logic, Bart Kosko*
Forms of reasoning

Generalized Modus Ponens:

Premise: \( x \text{ is } A' \)
Implication: if \( x \text{ is } A \) then \( y \text{ is } B \)
Consequence: \( y \text{ is } B' \)

Where \( A, A', B, B' \) are fuzzy sets and \( x \) and \( y \) are symbolic names for objects.

Forms of reasoning

Generalized Modus Tolens:

Premise: \( y \text{ is } B' \)
Implication: if \( x \text{ is } A \) then \( y \text{ is } B \)
Consequence: \( x \text{ is } A' \)

Where \( A, A', B, B' \) are fuzzy sets and \( x \) and \( y \) are symbolic names for objects.

Fuzzy rule as a relation

if \( x \text{ is } A \) then \( y \text{ is } B \)

“\( x \text{ is } A' \), “\( y \text{ is } B' \)” – fuzzy predicates \( A(x), B(y) \)

if \( A(x) \) then \( B(y) \) can be represented as a relation

\( R(x,y): A(x) \rightarrow B(y) \)

where \( R(x,y) \) can be considered a fuzzy set with 2-dimensional membership function

\( \mu_R(x,y) = \mu_A(x) \cdot \mu_B(y) \)

where \( \mu \) is fuzzy implication function

MIN fuzzy implication

* Interprets the fuzzy implication as the minimum operation [Mamdani].

\[
R_c = A \times B = \int_{x,y} \mu_A(x) \land \mu_B(y) \land (x,y)
\]

where \( \land \) is the min operator

PRODUCT fuzzy implication

* Interprets the fuzzy implication as the product operation [Larsen].

\[
R_p = A \times B = \int_{x,y} \mu_A(x) \cdot \mu_B(y) \land (x,y)
\]

where \( \cdot \) is the algebraic product operator

EXAMPLE OF FUZZY IMPLICATION

Fuzzy rule:

“If temperature is high, then humidity is fairly high”

Lets define:

* \( T \) – universe of discourse for temperature
* \( H \) – universe of discourse for humidity
* \( t \in T, h \in H \) – variables for temperature and humidity
* Denote “high” as \( A, A \subseteq T \)
* Denote “fairly high” as \( B, B \subseteq H \)

Then the rule becomes:

\( R(t,h): \text{if } t \text{ is } A \text{ then } h \text{ is } B \text{ or } R(t,h): R(t) \rightarrow R(h) \)
**EXAMPLE OF FUZZY IMPLICATION**

If we know A and B, we can find R(t,h) = A × B

<table>
<thead>
<tr>
<th>t</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ_A(t)</td>
<td>0.1</td>
<td>0.5</td>
<td>0.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>h</th>
<th>20</th>
<th>50</th>
<th>70</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ_B(h)</td>
<td>0.2</td>
<td>0.6</td>
<td>0.7</td>
<td>1</td>
</tr>
</tbody>
</table>

R_C(t, h) = A × B

= [μ_A(t) × μ_B(h)] / (t, h)

Mamdani (min) implication

**EXAMPLE OF FUZZY IMPLICATION**

We know R_C(t, h) for fuzzy rule

“If temperature is high, then humidity is fairly high”

According to this rule, what is the humidity when

“temperature is fairly high” or t is A’, A’∈T?

<table>
<thead>
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<th>t</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ_C(t)</td>
<td>0.01</td>
<td>0.25</td>
<td>0.6</td>
</tr>
</tbody>
</table>

**EXAMPLE OF FUZZY IMPLICATION**

We can use composition of fuzzy relations to find R(h)!

<table>
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<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ_C(t)</td>
<td>0.001</td>
<td>0.25</td>
<td>0.91</td>
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</table>

R(t)

R_C(t, h)

<table>
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<th>h</th>
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<td>μ_C(h)</td>
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</tr>
</tbody>
</table>

R(h) = R(t) o R_C(t, h)

**COMPOSITIONAL RULE OF INERENCE**

In order to draw conclusions from a set of rules (rule base) one needs a mechanism that can produce an output from a collection of rules. This is done using the compositional rule of inference.

Consider a single fuzzy rule and its inference

Rule: If v is A then w is C
Input: v is A'
Result: C'

A ⊂ U, C ⊂ W, v ⊂ U, and w ⊂ C.
The fuzzy rule is interpreted as an implication
R:A→C or R=A×C
When input A’ is given to the inference system, the output C’ = A’ o R

**COMPOSITIONAL RULE OF INERENCE**

C’ = A’ o R

“o” is the composition operator. The inference procedure is called “compositional rule of inference”. The inference mechanism is determined by two factors:

1. Implication operators:
   - Mamdani: min
   - Larsen: algebraic product
2. Composition operators:
   - Mamdani: max-min
   - Larsen: max-product

**COMPOSITIONAL RULE OF INERENCE**

Compositional rule of inference can be represented graphically as a combination of cylindrical extension, intersection and projection of fuzzy sets:

1. Build a cylindrical extension of A, A(x,y)
2. Determine intersection of R(x,y) and A(x,y)
3. Build projection of R(x,y)×A(x,y)
There are many methods to perform fuzzy inference. Consider a fuzzy rule:

\[ R_1: \text{if } u \text{ is } A_1 \text{ and } v \text{ is } B_1 \text{ then } w \text{ is } C_1 \]

Inputs \( u \) and \( v \) can be:

- crisp inputs. Crisp inputs can be treated as fuzzy singletons
- fuzzy sets \( A' \) and \( B' \)

**MAMDANI METHOD**

This method uses the minimum operation \( R_C \) as a fuzzy implication and the max-min operator for the composition.

Suppose a rule base is given in the following form:

\[ R_i: \text{if } u \text{ is } A_i \text{ and } v \text{ is } B_i \text{ then } w \text{ is } C_i, \quad i = 1, 2, \ldots, n \]

for \( u \in U, v \in V, \) and \( w \in W \).

Then, \( R_i = (A_i \text{ and } B_i) \rightarrow C_i \) is defined by

\[ \mu_{R_i} = \mu_{(A_i \text{ and } B_i) \rightarrow C_i}(u, v, w) \]

**INFECTION METHODS**

Case 1: Inputs are crisp and treated as fuzzy singletons.

\[ \mu_{C_i}(w) = [\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)] \rightarrow \mu_{C_i}(w) \]

Inference

Result

Example:

if temperature is high and humidity is high then fan speed is high

How to determine the fan speed for temperature 85°F and humidity 93%?

Mamdani method uses min operator \((\land)\) as fuzzy implication function \((\rightarrow)\):

\[ \mu_{C_i}(w) = \alpha \land \mu_{C_i}(w) \]

where \( \alpha = \mu_{A_i}(u_0) \land \mu_{B_i}(v_0) \)

\( \alpha \) is called “firing strength”, “matching degree”, “satisfaction degree”

For multiple rules (for example, two rules \( R_1 \) and \( R_2 \)):

\[ \mu_{C_i}(w) = \mu_{C_1}(w) \lor \mu_{C_2}(w) = [\alpha_1 \land \mu_{C_1}(w)] \lor [\alpha_2 \land \mu_{C_2}(w)] \]
**MAMDANI METHOD**

In general:
\[
\mu_c(w) = \max_{i=1}^{n} \left( \min \left( \mu_{A_i}(u), \mu_{B_i}(v) \right) \right) = \max_{i=1}^{n} \mu_{C_i}(w)
\]

**EXAMPLE OF MAMDANI METHOD**

Let the fuzzy rule base consist of one rule:

- **Input:** if \( u \) is \( A \) and \( v \) is \( B \) then \( w \) is \( C \), \( i = 1, 2, \ldots, n \)
- For \( u \in U, v \in V, w \in W \).

Then, \( R_i = (A_i \text{ and } B_i) \rightarrow C_i \) is defined by
\[
\mu_{R_i} = \mu_{A_i \text{ and } B_i \rightarrow C_i}(u, v, w)
\]

**EXAMPLE OF MAMDANI METHOD**

For multiple rules,
\[
\mu_{C_i}(w) = \frac{1}{n} \sum_{i=1}^{n} \left( \min(\mu_{A_i}(u), \mu_{B_i}(v)) \right) = \frac{1}{n} \mu_{C}(w)
\]

**EXAMPLE OF MAMDANI METHOD**

Case 2: Inputs are fuzzy sets \( A', B' \)

\[
\mu_{C_i}(w) = \alpha_i \land \mu_{C_i}(w)
\]

where \( \alpha_i = \min(\max(\mu_{A_i}(u), \mu_{B_i}(v)), \max(\mu_{A_i}(u), \mu_{B_i}(v))) \)

**LARSEN METHOD**

This method uses the product operation \( R_p \) as a fuzzy implication and the max-product operator for the composition.

Suppose a rule base is given in the following form:

- **Input:** if \( u \) is \( A \) and \( v \) is \( B \) then \( w \) is \( C \), \( i = 1, 2, \ldots, n \)
- for \( u \in U, v \in V, w \in W \).

Then, \( R_i = (A_i \text{ and } B_i) \rightarrow C_i \) is defined by
\[
\mu_{R_i} = \mu_{A_i \text{ and } B_i \rightarrow C_i}(u, v, w)
\]
LARSEN METHOD

Case 1: Inputs are crisp and treated as fuzzy singletons.
\[ u = u_0, \ v = v_0 \]
\[ \mu_C(w) = [\mu_A(u_0) \land \mu_B(v_0)] \rightarrow \mu_C(w) \]
Inference
\[ \text{result} = [\mu_A(u_0) \land \mu_B(v_0)] \land \mu_C(w) = \alpha_i \cdot \mu_C(w) \]
For multiple rules:
\[ C' = \bar{\bigsqcup}_i C_i' \]

Case 2: Inputs are fuzzy sets A', B'
\[ \mu_C(w) = \alpha_i \cdot \mu_C(w) \]
where \( \alpha_i = \min \{ \max(\mu_A(w) \land \mu_B(w)) \} \)
For multiple rules:
\[ C' = \hat{\bigsqcup}_i C_i' \]

EXAMPLE OF LARSEN METHOD

Let the fuzzy rule base consist of one rule:
R: If \( u \) is A and \( v \) is B then \( w \) is C
where A=(0, 2, 4), B=(3, 4, 5) and C=(3,4,5) are triangular fuzzy sets

Question 1: What is the output C' if the inputs are crisp values \( u_0=3, v_0=4 \)?

Question 2: What is the output C' if the inputs are fuzzy sets \( A'=(0, 1, 2) \) and \( B'=(2,3,4) \)?
DEFUZZIFICATION

◆ The output of Mamdani and Larsen inference methods is a fuzzy set!

◆ For practical applications a crisp value is often needed

◆ The process of converting a fuzzy answer into a crisp value is called defuzzification

SUMMARY

◆ Inference - the logical process by which new facts are derived from known facts by the application of inference rules.

◆ Fuzzy rules – a convenient way to represent knowledge

◆ A fuzzy rule can be represented as a fuzzy relation connected by a fuzzy implication function

◆ The fuzzy inference procedure is called the compositional rules of inference

SUMMARY

◆ Mamdani and Larsen methods are two very popular methods of fuzzy inference.

◆ There are many more inference methods that we will consider later!

◆ Defuzzification is needed for the results obtained through fuzzy inference.