# Specification of membership functions and Multi-dimensional fuzzy sets

## Fuzzy set definition

Definition: let X be a non-empty set and be called the universe of discourse. A fuzzy set $A \subset X$ is characterized by the membership function

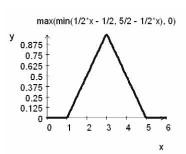
$$\mu_A : X \rightarrow [0,1]$$

where $\mu_A(x)$ is a grade (degree) of membership of x in set A.

## Typical membership functions

Triangular MF:

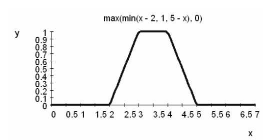
$$\text{triangularmf}(x, c, h) = \max\left(\min\left(\frac{h-c+x}{h}, \frac{c+h-x}{h}\right), 0\right)$$




## Typical membership functions

Trapezoidal MF:

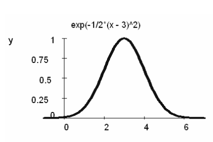
$$\text{trapmf}(x, a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$




## Typical membership functions

Gaussian MF:

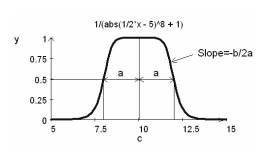
$$\text{gaussmf}(x, c, s) = e^{-\frac{(x-c)^2}{2 \cdot s^2}}$$




## Typical membership functions

Generalized bell MF:

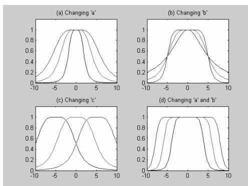
$$\text{gbellmf}(x, a, b, c) = \frac{1}{\left|\frac{x}{a} - \frac{c}{a}\right|^{2 \cdot b} + 1}$$



## Typical membership functions

Generalized bell MF:

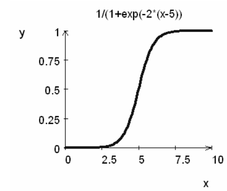
$$\mathrm{gbellmf}(x,\ a,\ b,\ c) = \frac{1}{\left|\frac{x}{a} - \frac{c}{a}\right|^{2 \cdot b} + 1}$$




## Typical membership functions

Sigmoidal MF:

$$\mathrm{sigmf}(x,\ a,\ c) = \frac{1}{e^{-a(x-c)} + 1}$$




## Typical membership functions

The list of MFs introduced here is by no means exhaustive

Other specialized MFs can be created for specific applications if necessary

Any type of continuous probability distribution functions can be used as an MF

## Multi-dimensional fuzzy sets

Increase in dimensions:

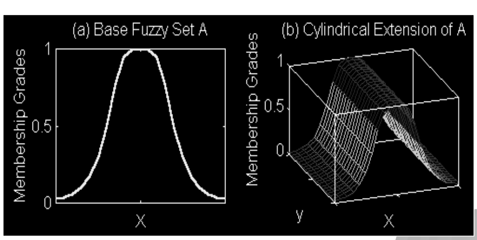
A n-dimensional fuzzy set can be extended to n+1 dimensions through cylindrical extension:

$\mu_B\ (a,\ b) = \mu_A\ (a)$

$a \in A,\ b \in B$

## Multi-dimensional fuzzy sets

**Base set A** **Cylindrical Ext. of A**




## Multi-dimensional fuzzy sets

Decrease in dimensions:

A n-dimensional fuzzy set can be decreased to k=n-1 dimensions through projection:

$$\mu_{R_{X_{i1} \times X_{i2} \times \ldots \times X_{ik}}}(x_{i1}, x_{i2}, \ldots, x_{ik}) = \max_{X_{j1}, X_{j2}, \ldots, X_{jn}} \mu_R(x_1, x_2, \ldots, x_n)$$

# Multi-dimensional fuzzy sets

Example 2-D to 1-D:

Let R be a two-dimensional fuzzy set on X*Y. Then the projections of R onto X and Y are defined as:

$$R_X = \int_X \left[ \max_y \mu_R(x,y) \right] | x$$

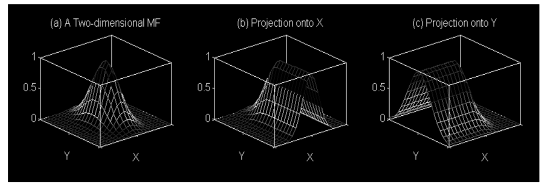
and

$$R_Y = \int_Y \left[ \max_x \mu_R(x,y) \right] | y$$

respectively.

# Multi-dimensional fuzzy sets

**Two-dimensional MF** | **Projection onto X** | **Projection onto Y**




Note that projections are 1-dimensional!

2-D shapes are shown for illustration purposes only.

# Multi-dimensional fuzzy sets

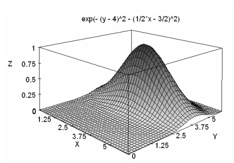
Composite & non-composite MFs :

Suppose that the fuzzy A = "(x,y) is near (3,4)" is defined by:

$$f(x,\ y) = e^{-(y-4)^2 - \left(\frac{x}{2} - \frac{3}{2}\right)^2}$$

or

$$f(x,\ y) = e^{-(y-4)^2} \cdot e^{-\left(\frac{x}{2} - \frac{3}{2}\right)^2}$$




# Multi-dimensional fuzzy sets

A = "(x,y) is near (3,4)" $f(x,\ y) = e^{-(y-4)^2} \cdot e^{-\left(\frac{x}{2} - \frac{3}{2}\right)^2}$

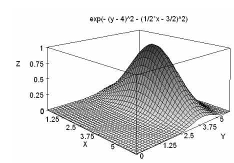
is a composition (intersection) of two membership functions

$$fx(x,\ y) = e^{-\left(\frac{x}{2} - \frac{3}{2}\right)^2}$$

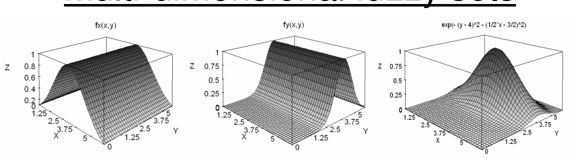
"x is near 3"

$$fy(x,\ y) = e^{-(y-4)^2}$$

"y is near 4"



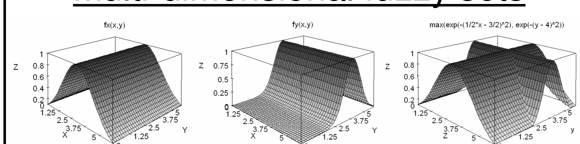

# Multi-dimensional fuzzy sets




$fx(x,\ y) = e^{-\left(\frac{x}{2} - \frac{3}{2}\right)^2}$ $fy(x,\ y) = e^{-(y-4)^2}$ $f(x,\ y) = e^{-(y-4)^2} \cdot e^{-\left(\frac{x}{2} - \frac{3}{2}\right)^2}$

A composite two-dimensional MF is usually the result of two one-dimensional membership functions joined by union or intersection

# Multi-dimensional fuzzy sets

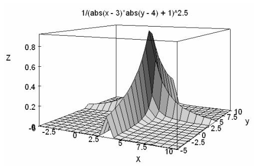



$fx(x,\ y) = e^{-\left(\frac{x}{2} - \frac{3}{2}\right)^2}$ $fy(x,\ y) = e^{-(y-4)^2}$ $f(x, y) = fx(x, y) \cup fy(x, y)$

The same example with two functions joined by standard union.

## Multi-dimensional fuzzy sets

The following function cannot be decomposed into two component function, therefore it is non-composite:




$$f(x, y) = (|x-3| \cdot |y-4| + 1)^{-2.5}$$