

Basic Concepts of Fuzzy Logic

- Apparatus of fuzzy logic is built on:

⇒ Fuzzy sets: describe the value of variables

Possibility distributions: constraints on the value of a variable

Linguistic variables: qualitatively and quantitatively described by fuzzy sets

Fuzzy if-then rules: knowledge

*Fuzzy Logic: Intelligence, Control, and Information - J. Yen and R. Langari, Prentice Hall 1999

Fuzzy sets

A fuzzy set is a set with a smooth boundary.

An element of a fuzzy set can belong to that set partially to a degree and the set does not have crisp boundaries.

A fuzzy set is defined by a functions that maps objects in a domain of concern into their membership value in a set.

Such a function is called the *membership function*.

Fuzzy sets

Definition: let X be a non-empty set and be called the universe of discourse. A fuzzy set $A \subset X$ is characterized by the membership function

$$\mu_A : X \rightarrow [0,1]$$

where $\mu_A(x)$ is a grade (degree) of membership of x in set A .

Fuzzy sets

$$\mu_A : X \rightarrow [0,1]$$

Note : since $\{0,1\} \in [0,1]$
all crisp sets are fuzzy sets!

Fuzzy sets

Definition of fuzzy sets:

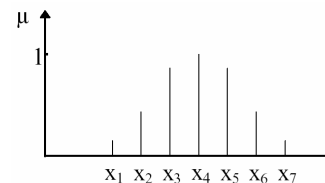
Fuzzy set A can be represented as a set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

Fuzzy sets

Discrete example:

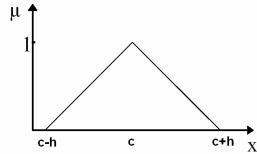
$$\mu_A = 0.1/x_1 + 0.4/x_2 + 0.8/x_3 + 1.0/x_4 + 0.8/x_5 + 0.4/x_6 + 0.1/x_7$$



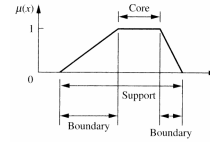
Fuzzy sets

Continuous example:

$$\mu_A(x) = \begin{cases} 1 + \frac{x-c}{h}, & x \in [c-h, c] \\ 1 - \frac{x-c}{h}, & x \in [c, c+h] \\ 0, & \text{otherwise} \end{cases}$$



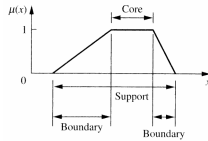
Properties of fuzzy sets



Support : support of a fuzzy set A is a crisp set that contains all elements of A with non-zero membership grade:

$$\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$$

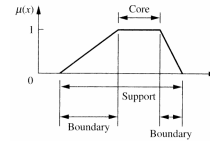
Properties of fuzzy sets



Core: comprises those elements x of the universe such that $\mu_A(x) = 1$.

$$\text{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$$

Properties of fuzzy sets



Boundary : boundaries comprise those elements x of the universe such that $0 < \mu_A(x) < 1$

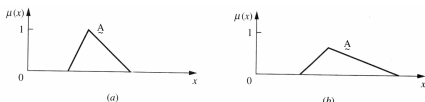
$$\text{bnd}(A) = \{x \in X \mid 0 < \mu_A(x) < 1\}$$

Properties of fuzzy sets

Height : the height of a fuzzy set A is defined as

$$\text{hgt}(A) = \sup_{x \in X} \mu_A(x)$$

Set A is called normal if $\text{hgt}(A)=1$ and subnormal if $\text{hgt}(A)<1$



Fuzzy sets that are normal (a) and subnormal (b).

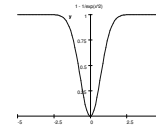
Properties of fuzzy sets

Question?

Is the fuzzy set defined as

$$\mu_A(x) = 1 - 1/e^{-x^2}$$

Normal or subnormal?

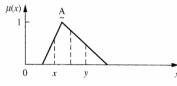


Properties of fuzzy sets

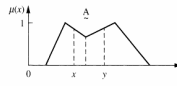
Convex Fuzzy set: a fuzzy set A is convex iff

$$\forall x, y \in X \text{ and } \forall \lambda \in [0,1]$$

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$



Convex, normal fuzzy set

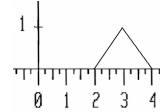


Non convex, normal fuzzy set

Properties of fuzzy sets

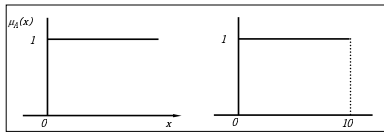
Fuzzy number: a fuzzy set A is a fuzzy number if the fuzzy set is

- Convex
- Normal
- The core consists of one value only
- MF is piecewise continuous



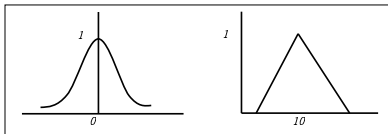
Example: fuzzy 3

Properties of fuzzy sets



Set "positive number"

Set "positive number not exceeding 10"



Set "number near 0"

Set "number near 10"

Operations on fuzzy sets

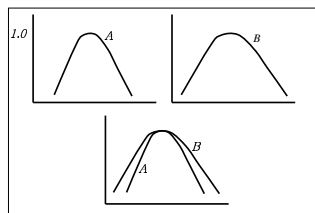
1. Empty set
 $\mu_{\emptyset} \equiv 0$

2. Basic set (universe)
 $\mu_X \equiv 1$

3. Identity
 $A = B \Leftrightarrow \mu_A(x) = \mu_B(x) \quad \forall x \in X$

Operations on fuzzy sets

4. Subset
 $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \quad \forall x \in X$



Operations on fuzzy sets

5. Union

Axioms for union function

$$U : [0,1] \times [0,1] \rightarrow [0,1] \quad \mu_{A \cup B}(x) = U[\mu_A(x), \mu_B(x)]$$

- $U(0,0) = 0, U(0,1) = 1, U(1,0) = 1, U(1,1) = 1$
- $U(a,b) = U(b,a)$ (Commutativity)
- If $a \leq a'$ and $b \leq b'$, $U(a, b) \leq U(a', b')$ (monotonicity).
- $U(U(a, b), c) = U(a, U(b, c))$ (Associativity)
- Function U is continuous.
- $U(a, a) = a$ (idempotency)

Fuzzy advice



Do not sleep in class!!!

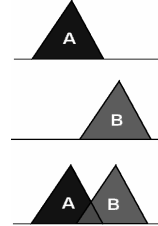
For sleep-deprived: *SL119: Linear Sleep*
Advanced class in mattress manipulation.
Topics include unconsciousness and hibernation.

Operations on fuzzy sets

5. Union

$$U[\mu_A, \mu_B] = \max[\mu_A, \mu_B],$$

$$\forall x \in X : \mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$



Operations on fuzzy sets

6. Intersection

Axioms for intersection function
 $I: [0, 1] \times [0, 1] \rightarrow [0, 1]$ $\mu_{A \cap B}(x) = I(\mu_A(x), \mu_B(x))$

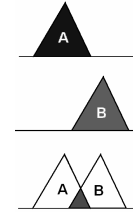
- $I(1, 1) = 1, I(1, 0) = 0, I(0, 1) = 0, I(0, 0) = 0$
- $I(a, b) = I(b, a)$, Commutativity.
- If $a \leq a'$ and $b \leq b'$, $I(a, b) \leq I(a', b')$, monotonicity.
- $I(I(a, b), c) = I(a, I(b, c))$, Associativity.
- I is a continuous function.
- $I(a, a) = a$, idempotency.

Operations on fuzzy sets

6. Intersection

$$I[\mu_A, \mu_B] = \min[\mu_A, \mu_B]$$

$$\forall x \in X : \mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$



Operations on fuzzy sets

7. Complement

Axioms for Complement function
 $C: [0, 1] \rightarrow [0, 1]$

- Boundary conditions $C(0) = 1, C(1) = 0$
- C is monotonic non-increasing
 $a, b \in [0, 1]$ if $a < b$, then $C(a) \geq C(b)$
- C is a continuous function.
- C is involutive $C(C(a)) = a$ for all $a \in [0, 1]$

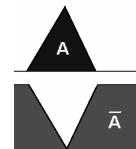
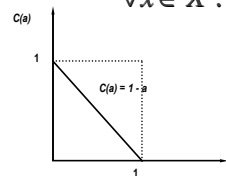
Operations on fuzzy sets

7. Complement

Standard complement function:

$$C(a) = 1 - a$$

$$\forall x \in X : \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$



Operations on fuzzy sets

7. Complement

Continuous complement function:

$$C(a) = 1/2(1 + \cos(\pi a))$$

$$\forall x \in X : \mu_{\bar{A}}(x) = 1/2(1 + \cos(\pi \mu_A(x)))$$



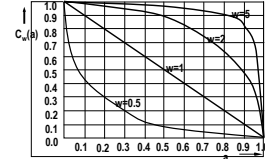
Operations on fuzzy sets

7. Complement

Yager complement function:

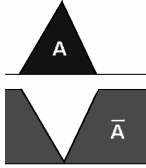
$$C_w(a) = (1 - a^w)^{1/w}, \quad w \in (-1, \infty)$$

$$\forall x \in X : \mu_{\bar{A}}(x) = (1 - \mu_A(x)^w)^{1/w}$$



Operations on fuzzy sets

7. Complement



Note: the laws of excluded middle

$$A \cap \bar{A} = \emptyset$$

and the law of contradiction

$$A \cup \bar{A} = X$$

Are not valid for fuzzy sets!

Operations on fuzzy sets

Properties of fuzzy operations:

- | | |
|--------------------|--|
| (1) Involution | $\bar{\bar{A}} = A$ |
| (2) Commutativity | $A \cup B = B \cup A$
$A \cap B = B \cap A$ |
| (3) Associativity | $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$ |
| (4) Distributivity | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ |
| (5) Idempotency | $A \cup A = A$
$A \cap A = A$ |
| (6) Absorption | $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$ |

Operations on fuzzy sets

Properties of fuzzy operations:

- | | |
|---------------------------------------|--|
| (7) Absorption by X and \emptyset | $A \cup X = X$
$A \cap \emptyset = \emptyset$ |
| (8) Identity | $A \cup \emptyset = A$
$A \cap X = A$ |
| (9) De Morgan's law | $\overline{A \cap B} = \bar{A} \cup \bar{B}$
$\overline{A \cup B} = \bar{A} \cap \bar{B}$ |
| (10) Equivalence formula | $(\bar{A} \cup B) \cap (A \cup \bar{B}) = (\bar{A} \cap \bar{B}) \cup (A \cap B)$ |
| (11) Symmetrical difference formula | $(\bar{A} \cap B) \cup (A \cap \bar{B}) = (\bar{A} \cup \bar{B}) \cap (A \cup B)$ |

Operations on fuzzy sets

Power of fuzzy sets:

$$\mu_{A^2}(x) = [\mu_A(x)]^2, \quad \forall x \in X$$

$$\mu_{A^m}(x) = [\mu_A(x)]^m, \quad \forall x \in X^m$$

Operations on fuzzy sets

α -cut set

- $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$, α is an arbitrary real number in $[0,1]$
- α -cut set is a crisp set

